# IRREDUCIBLE COMPONENTS OF CHARACTERISTIC VARIETIES

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Abstract. We give a dimension bound on the irreducible components of the characteristic variety of a system of linear partial differential equations defined from a suitable filtration of the Weyl algebra $A_n(k)$. This generalizes an important consequence of the fact that a characteristic variety defined from the order filtration is involutive.

More explicitly, we consider a filtration of $A_n(k)$ induced by any vector $(\mathbf{u}, \mathbf{v}) \in \mathbb{Z}^n \times \mathbb{Z}^n$ such that the associated graded algebra is the commutative polynomial ring in $2n$ indeterminates. The order filtration is the special case $(\mathbf{u}, \mathbf{v}) = (\mathbf{0}, \mathbf{1})$. Any finitely generated left $A_n(k)$-module $M$ has a good filtration with respect to $(\mathbf{u}, \mathbf{v})$ and this gives rise to a characteristic variety $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ which depends only on $(\mathbf{u}, \mathbf{v})$ and $M$. When $(\mathbf{u}, \mathbf{v}) = (\mathbf{0}, \mathbf{1})$, the characteristic variety is involutive and this implies that its irreducible components have dimension at least $n$. In general, the characteristic variety may fail to be involutive, but we are still able to prove that each irreducible component of $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ has dimension at least $n$.

## 1. Introduction

The geometry of the characteristic variety plays a central role in the study of systems of linear partial differential equations. In algebraic analysis, the characteristic variety of a linear system is obtained from a filtration of the corresponding $\mathcal{D}$-module. When $\mathcal{D}$ is equipped with the order filtration, Sato, Kashiwara and Kawai [SKK] and Gabber [Gab] show that the characteristic variety of any $\mathcal{D}$-module is involutive with respect to the natural symplectic structure on the cotangent bundle. As a consequence, they deduce the "Strong Fundamental Theorem of Algebraic Analysis", which says that, if $\mathcal{D}$ is the sheaf of differential operators on an $n$-dimensional variety, then each irreducible component of the characteristic variety must have dimension at least that $n$. In this paper, we work over a field $k$ and consider left $A_n(k)$-modules which correspond differential operators on $\mathbb{A}^n_k$ with polynomial coefficients. Our goal is to extend this dimension bound to all filtrations of the Weyl algebra $A_n(k)$ for which the associated graded ring is a commutative

polynomial ring in $2n$ variables and to extend this assertion to a larger class of algebras.

We are primarily interested in this larger class of filtrations because of its connection with monomial ideals in a commutative polynomial ring. Specifically, for a generic vectors $(\mathbf{u}, \mathbf{v})$, the characteristic variety $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ is given by a square-free monomial ideal or equivalently a simplicial complex. Monomials ideals form an important link between algebraic geometry, combinatorics and commutative algebra. Much of the success of Gröbner bases theory comes from an understanding of monomial ideals. We believe that further exploration of this connection will lead to new insights into $A_n(k)$-modules and, perhaps, primitive ideals. Problems of making effective computations in algebraic analysis, provide a secondary motivation for considering filtrations other than the standard or order filtration. The choice of filtration can significantly effect the complexity of the characteristic variety.

To state our theorems more explicitly, we introduce some notation. We write $x_1, \dots, x_n, y_1, \dots, y_n$ for the generators of $A_n(k)$ satisfying the relations $x_i x_j - x_j x_i = 0$, $y_i y_j - y_j y_i = 0$ and $y_i x_j - x_j y_i = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker symbol. Each vector $(\mathbf{u}, \mathbf{v}) \in \mathbb{Z}^n \times \mathbb{Z}^n$ induces an increasing filtration on $A_n(k)$ by setting $\deg(x_i) = u_i$ and $\deg(y_i) = v_i$. We shall focus those vectors $(\mathbf{u}, \mathbf{v})$ for which the associated graded ring $\mathrm{gr}_{(\mathbf{u},\mathbf{v})}(A_n(k))$ is the commutative polynomial ring $S = k[\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_n]$. Analogously, for any finitely generated left $A_n(k)$-module $M$, we can filter $M$ by assigning degrees to a generating set of $M$. The associated graded module $\mathrm{gr}(M)$ is then a module over the polynomial ring $S$ and, hence, the prime radical of the annihilator of $\mathrm{gr}(M)$ defines a variety in $\mathbb{A}_k^{2n}$. This variety is called the characteristic variety $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ of $M$. It is independent of the choice of degrees and generators for $M$, but depends on the filtration of $A_n(k)$. The main result of this paper is the following:

**Theorem 1.1.** *Let $k$ be a field of characteristic zero and let $M$ be a finitely generated left $A_n(k)$-module. If the integer vector $(\mathbf{u}, \mathbf{v})$ induces a filtration satisfying $\mathrm{gr}_{(\mathbf{u},\mathbf{v})}(A_n(k)) = S$, then every irreducible component of $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ has dimension at least $n$.*

The conclusion of this theorem is vacuously satisfied when $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ is empty and this can occur as the following example illustrates: if $M = A_2(k)/A_2(k) \cdot I$, where $I$ is the left ideal $\langle y_1 - 1, y_2 - 1 \rangle$, then $\mathrm{Ch}_{(\mathbf{2},-\mathbf{1})}(M)$ corresponds to the $S$-ideal $\langle 1 \rangle$ indicating that the characteristic variety is empty. However, for $(\mathbf{u}, \mathbf{v})$ nonnegative and $M \neq 0$, the characteristic variety $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ is never empty.

Theorem 1.1 refines Bernstein's inequality [Bor] which says that there exists an irreducible component of $\mathrm{Ch}_{(\mathbf{1},\mathbf{1})}(M)$ of dimension at least $n$. On the other hand, for the order filtration $(\mathbf{u}, \mathbf{v}) = (\mathbf{0}, \mathbf{1})$, Theorem 1.1 follows from the fact that the characteristic variety $\mathrm{Ch}_{(\mathbf{0},\mathbf{1})}(M)$ is involutive with respect to the natural symplectic structure on $\mathbb{A}_k^{2n}$. The involutivity of $\mathrm{Ch}_{(\mathbf{0},\mathbf{1})}(M)$ was first established by Sato, Kashiwara and Kawai [SKK] using mirco-local analysis; Gabber [Gab] provided a purely algebraic proof. In our more general case, a different proof is necessary because the characteristic variety $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ is not always involutive under the natural symplectic structure on $\mathbb{A}_k^{2n}$. For example, the characteristic variety of $A_2(k)/A_2(k) \cdot I$, where $I$ is the left ideal $\langle y_1^2 - y_2, x_1y_1 + 2x_2y_2 \rangle$, with respect to the vector $(1, 1, 1, 3)$ is given by the non-involutive $S$-ideal $\langle \bar{x}_2, \bar{y}_2 \rangle \cap \langle \bar{y}_1, \bar{y}_2 \rangle$.

The general techniques used in the proof of Theorem 1.1 apply to a larger collection of $k$-algebras. We develop these methods for a skew polynomial ring $R$ which is an almost centralizing extension of a commutative polynomial ring (see section 2 for a precise definition). We write GKdim for the Gelfand-Kirillov dimension. The second major result of this paper is the following:

**Theorem 1.2.** *Let $(\mathbf{u}, \mathbf{v})$ be an integer vector which induces an increasing filtration on $R$ such that $\mathrm{gr}_{(\mathbf{u},\mathbf{v})}(R)$ is a commutative polynomial ring. If $M$ is a finitely generated left $R$-module such that, for every nonzero submodule $M'$ of $M$, one has $\mathrm{GKdim}\, M' \geq p$, then every irreducible component of $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ has dimension at least $p$.*

In particular, this theorem says that if $M$ is GKdim-pure, that is $\mathrm{GKdim}\, M' = \mathrm{GKdim}\, M$ for every nonzero submodule $M'$ of $M$, then the characteristic variety is equidimensional.

Theorem 1.2 generalizes known equidimensionality results to a larger class of filtrations. In particular, when $R$ is the enveloping algebra of finite dimensional Lie algebra, it extends Gabber's equidimensionality theorem [GL, Théorème 1] beyond the standard filtration. For certain skew polynomial rings, it also extends the equidimensionality theorem in Huishi and van Oystaeyen [HvO, Corollary III 4.3.6] to non-Zariskian filtrations. Specifically, when the vector $(\mathbf{u}, \mathbf{v})$ has a negative entry, the induced filtration is not Zariskian. Our proof of Theorem 1.2 involves studying the growth of filtered modules and Gröbner basis theory and differs significantly from the homological methods used by Björk [Bj1], Gabber [GL] and Huishi and van Oystaeyen [HvO].

We now describe the contents of this paper. In the next section, we list the global notation and our general conventions. In particular, we recall the definitions of an almost centralizing extension, an

increasing filtration, an associated Rees ring and a characteristic variety. We also introduce the polynomial region, the space of vectors inducing filtrations for which the associated graded algebra is the commutative polynomial ring. In Section 3, we use properties of Rees modules to connect irreducible components of the characteristic variety to the Gelfand-Kirillov dimension of submodules. To guarantee that the Gelfand-Kirillov dimension is well behaved, we restrict our attention to finite dimensional filtrations throughout this section. Section 4 develops the Gröbner basis theory for our skew polynomial ring. These tools are applied in Section 5 to construct a combinatorial object, called the Gröbner fan. This generalizes the Gröbner fan of Mora and Robbiano [MR] in the case of commutative polynomial rings and Assi, Castro-Jiménez and Granger [ACG] in the case of the Weyl algebra. In the last section, we use the use the Gröbner fan to extend our results for finite dimensional filtrations to all filtrations in the polynomial region and prove our main theorems.

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## 2. Rings, Filtrations and Modules

In this section, we recall the notion of an almost centralizing extension and list some of its basic properties. Throughout this paper $k$ denotes a field.

**Almost centralizing extensions.** Let $B$ be the commutative polynomial ring $k[x_1, \ldots, x_m]$. We concentrate on a $k$-algebra $R$ which is generated by $x_1, \ldots, x_m, y_1, \ldots, y_n$ subject only to the relations:

$$(R1) \qquad y_i x_j - x_j y_i = Q^1_{i,j}(x)\,;$$

$$(R2) \qquad y_i y_j - y_j y_i = Q^2_{i,j}(x,y) = Q^{2,0}_{i,j}(x) + \sum_{\ell=1}^{n} Q^{2,\ell}_{i,j}(x) y_\ell$$

where $Q^1_{i,j}(x)$, $Q^{2,\ell}_{i,j}(x) \in B$ for all $1 \leq \ell \leq n$. The skew polynomial ring $R$ is called an almost centralizing extension of $B$. The Poincaré-Birkhoff-Witt theorem generalizes to $R$ and, hence, the set of standard monomial $\{x^{\mathbf{a}} y^{\mathbf{b}} = x_1^{a_1} \cdots x_m^{a_m} y_1^{b_1} \cdots y_n^{b_1} : (\mathbf{a}, \mathbf{b}) \in \mathbb{N}^m \times \mathbb{N}^n\}$ forms a $k$-basis. In particular, each element $f \in R$ has a unique standard expression of the form $\sum \kappa_{\mathbf{a},\mathbf{b}} x^{\mathbf{a}} y^{\mathbf{b}}$. For more information about almost centralizing extension see subsections 8.6.6 and 8.6.7 in McConnell and Robson [McR].

*Example* 2.1. If $\mathfrak{g}$ is a finite dimensional Lie algebra over $k$ then any crossed product $B * U(\mathfrak{g})$ is an almost centralizing extension of $B$. Notably, the polynomial ring in $m+n$ central indeterminates over $k$, the Weyl algebra $A_n(k)$ and the universal enveloping algebra $U(\mathfrak{g})$ all have this form.

**The polynomial region.** A vector $(\mathbf{u},\mathbf{v}) \in \mathbb{R}^m \times \mathbb{R}^n$ induces an increasing filtration of $R$ as follows: for each $i \in \mathbb{Z}$, consider the vector space $F_iR := k \cdot \left\{ x^{\mathbf{a}}y^{\mathbf{b}} : \lceil \mathbf{u} \rceil \cdot \mathbf{a} + \lceil \mathbf{v} \rceil \cdot \mathbf{b} \leq i,\, \mathbf{a} \in \mathbb{N}^m,\, \mathbf{b} \in \mathbb{N}^n \right\}$ , where $\lceil \mathbf{u} \rceil = (\lceil u_1 \rceil, \dots, \lceil u_m \rceil)$ and $\lceil u_i \rceil$ is the smallest integer greater than $u_i$. This clearly gives an increasing sequence of subspaces satisfying the conditions $1 \in F_0R$ and $\bigcup_{i\in\mathbb{Z}} F_iR = R$. When $F_iR \cdot F_jR \subseteq F_{i+j}R$, the associated graded ring is $\operatorname{gr}_{(\mathbf{u},\mathbf{v})}(R) = \bigoplus_{i\in\mathbb{Z}} F_iR/F_{i-1}R$. An element $f$ belonging to the vector space $F_iR - F_{i-1}R$ is said to have degree $i$ and we write $\deg_{(\mathbf{u},\mathbf{v})} f = i$.

The polynomial region associated to $R$, denoted $\mathrm{PR}(R)$, is set of all real vectors $(\mathbf{u},\mathbf{v})$ such that $\operatorname{gr}_{(\mathbf{u},\mathbf{v})}(R)$ is the commutative polynomial ring generated by the initial forms $\bar{x}_i$ and $\bar{y}_i$ of $x_i$ and $y_i$. We denote this commutative polynomial ring by $S$. Since $S$ is noetherian ring, $R$ is both a left and a right noetherian ring. We will focus exclusively on vectors $(\mathbf{u},\mathbf{v})$ belonging to the polynomial region. The next proposition provides a more explicit interpretation of $\mathrm{PR}(R)$.

**Proposition 2.2.** *The polynomial region* $\mathrm{PR}(R)$ *is the open convex polyhedral cone in* $\mathbb{R}^m \times \mathbb{R}^n$ *given by the intersection of the following open half-spaces:*

$$\begin{array}{rcll} \deg_{(\mathbf{u},\mathbf{v})} x_iy_j & > & \deg_{(\mathbf{u},\mathbf{v})} Q^1_{i,j} & \text{for all } 1 \leq j \leq n \text{ and } 1 \leq i \leq m, \\ \deg_{(\mathbf{u},\mathbf{v})} y_iy_j & > & \deg_{(\mathbf{u},\mathbf{v})} Q^2_{i,j} & \text{for all } 1 \leq i,j \leq n. \end{array}$$

*Proof.* Without loss of generality, we may replace $(\mathbf{u},\mathbf{v})$ with $(\lceil \mathbf{u} \rceil, \lceil \mathbf{v} \rceil)$. Let $f_1, \dots, f_j$ be elements of the $k$-vector space generated by the elements $x_1, \dots, x_m, y_1, \dots, y_n$. Now, if $\sigma$ is a permutation of $\{1, \dots, j\}$, we claim that $f_1f_2\cdots f_j \in f_{\sigma(1)}f_{\sigma(2)}\cdots f_{\sigma(j)} + F_{\ell-1}R$, where $\ell$ is the degree of the left hand side. Since the elements $x_1, \dots, x_m$ commute and $Q^1_{i,j}(x)$ and $Q^2_{i,j}(x,y)$ are at most linear in the variables $y_1, \dots, y_n$, it suffices to prove this when $\sigma$ is a transposition. By linearity, the assertion is equivalent to the conditions:

$$\begin{aligned} y_ix_j - x_jy_i &= Q^1_{i,j}(x) \in F_{u_i+u_j-1}R \\ y_iy_j - y_jy_i &= Q^2_{i,j}(x,y) \in F_{v_i+v_j-1}R\,. \end{aligned}$$

We conclude that $F_iR \cdot F_jR \subseteq F_{i+j}R$ and that $\operatorname{gr}_{(\mathbf{u},\mathbf{v})}(R)$ is a commutative $k$-algebra generated by $\bar{x}_1, \dots, \bar{x}_m, \bar{y}_1, \dots, \bar{y}_n$.

To see that $\mathrm{gr}_{(\mathbf{u},\mathbf{v})}(R) = S$, it suffices to see that there are no $k$-linear relations between the monomials in $\mathrm{gr}_{(\mathbf{u},\mathbf{v})}(R)$. A relation among the monomials in $\mathrm{gr}_{(\mathbf{u},\mathbf{v})}(R)$, it would yield a relation among the standard monomials in $R$. However, the standard monomials form a $k$-basis for $R$ which completes the proof. $\square$

*Remark* 2.3. If $p = \max\{\deg_{(\mathbf{1},\mathbf{0})} Q_{i,j}^{\ell} \ : \ \text{for all } i,\, j,\, \ell\} + 1$, then the positive vector $(\mathbf{1}, p\mathbf{1})$ belongs to the polynomial region $\mathrm{PR}(R)$.

*Example* 2.4. The polynomial region for $S$ is the entire space $\mathbb{R}^m \times \mathbb{R}^n$ and $\mathrm{PR}(A_n(k)) = \{(\mathbf{u},\mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n : u_i + v_i > 0 \text{ for all } 1 \leq i \leq n\}$. If $\mathfrak{sl}_2(k)$ has the standard basis $y_1, y_2, y_3$ such that $y_2y_3 - y_3y_2 = 2y_3$, $y_2y_1 - y_1y_2 = -2y_1$ and $y_1y_3 - y_3y_1 = y_2$, then $\mathrm{PR}(U(\mathfrak{sl}_2(k)))$ is the open cone in $\mathbb{R}^3$ given by the inequalities $v_1 + v_3 > v_2$ and $v_2 > 0$.

**Rees rings.** The associated Rees ring of $R$ with respect to $(\mathbf{u},\mathbf{v})$ is the graded $k$-module $\widetilde{R} = \bigoplus_{i \in \mathbb{Z}} F_iR$. The $k$-algebra structure on $R$ makes $\widetilde{R}$ into a graded $k$-algebra. For $f \in F_iR$, we write the homogeneous element represented by $f$ in $\widetilde{R}_i$ as $(\tilde{f})_i$. Observe that $(\tilde{1})_1 \in \widetilde{R}_1$ is a central nonzero-divisor. We denote this canonical element by $x_0$. More concretely, the Rees ring $\widetilde{R}$ is generated by $x_0, \dots, x_m, y_1, \dots, y_n$ subject to the relations:

$$(\widetilde{R}1) \qquad y_ix_j - x_jy_i = x_0^{\lceil u_j \rceil + \lceil v_i \rceil - q_{i,j}^1} Q_{i,j}^1(x);$$

$$(\widetilde{R}2) \qquad y_iy_j - y_jy_i = x_0^{\lceil v_i \rceil + \lceil v_j \rceil - q_{i,j}^2} Q_{i,j}^2(x,y),$$

where $Q_{0,j}^1(x) = 0$ for all $1 \leq j \leq n$ and $q_{i,j}^{\ell} = \deg_{(\mathbf{u},\mathbf{v})} Q_{i,j}^{\ell}$ for $1 \leq \ell \leq 2$. We stress that $\widetilde{R}$ is an almost centralizing extension of $B[x_0]$ and the relations $(\widetilde{R}1)$ and $(\widetilde{R}2)$ are homogeneous with respect to $(\mathbf{u},\mathbf{v})$. The condition that $(\mathbf{u},\mathbf{v})$ belongs to the polynomial region $\mathrm{PR}(R)$ insures that $x_0$ has a nonnegative exponent in relations $(\widetilde{R}1)$ and $(\widetilde{R}2)$. The homogenization map from $R$ to $\widetilde{R}$ is defined as follows: for $f = \sum_{\mathbf{a},\mathbf{b}} \kappa_{\mathbf{a},\mathbf{b}} x^{\mathbf{a}} y^{\mathbf{b}}$ in $R$, we set $\tilde{f} = \sum_{\mathbf{a},\mathbf{b}} \kappa_{\mathbf{a},\mathbf{b}} x_0^{i - \mathbf{u}\cdot\mathbf{a} - \mathbf{v}\cdot\mathbf{b}} x^{\mathbf{a}} y^{\mathbf{b}}$ where $\deg_{(\mathbf{u},\mathbf{v})}(f) = i$. In the other direction, the substitution $x_0 = 1$ gives a $k$-algebra homomorphism $\widetilde{R}$ to $R$.

**Filtered modules and characteristic varieties.** All modules and ideals considered in this paper will be finitely generated left modules and left ideals respectively. We write $M$ for a finitely generated $R$-module. By a filtered $R$-module, we mean that there is an increasing sequence of $k$-vector spaces $F_iM$ for $i \in \mathbb{Z}$ satisfying the conditions: $F_iR \cdot F_jM \subseteq F_{i+j}M$, and $\bigcup_{i \in \mathbb{Z}} F_iM = M$. We define the associated

graded module to be $\operatorname{gr}(M) = \bigoplus_{i\in\mathbb{Z}} F_iM/F_{i-1}M$. It follows that $\operatorname{gr}(M)$ is a graded $\operatorname{gr}_{(\mathbf{u},\mathbf{v})}(R)$-module.

A good filtration is a filtration of an $R$-module $M$ for which there exists elements $z_1, \dots, z_p$ in $M$ and integers $w_1, \dots, w_p$ such that $F_iM = \sum_{j=1}^{p} F_{i-w_j}R \cdot z_j$. Every finitely generated $R$-module $M$ has a good filtration and, conversely, any module with a good filtration is necessarily finitely generated over $R$. For a good filtration of $M$, we define the characteristic ideal $I(M)$ to be the prime radical of $\operatorname{Ann}_S(\operatorname{gr}(M))$. Since any two good filtrations are equivalent, the characteristic ideal $I(M)$ is independent of the choice of good filtration; however $I(M)$ does depend on $(\mathbf{u},\mathbf{v})$. The characteristic variety of $M$ is the reduced scheme $\operatorname{Ch}_{(\mathbf{u},\mathbf{v})}(M) = \operatorname{Spec}(\operatorname{gr}_{(\mathbf{u},\mathbf{v})}(R)/I(M))$.

For a filtered $R$-module $M$, we define the associated Rees module to be the graded $k$-module $\widetilde{M} = \bigoplus_{i\in\mathbb{Z}} F_iM$. The $R$-module structure on $M$ makes $\widetilde{M}$ into a graded $\widetilde{R}$-algebra. More details on Rees modules can be found in section I.4 of Huishi and van Oystaeyen [HvO].

## 3. Finite Dimensional Filtrations

Under the assumption that $R$ has a finite dimensional filtration, we are able to relate the dimension of the irreducible components of $\operatorname{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ to the Gelfand-Kirillov dimension of submodules of $M$. We accomplish this by using the Rees module $\widetilde{M}$ to link submodules of $M$ and graded submodules of $\operatorname{gr}(M)$. We begin with a brief discussion of Gelfand-Kirillov dimension.

We define the Gelfand-Kirillov dimension only for $R$-modules with a given finite dimensional filtration. Recall that a function $\phi\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ has polynomial growth, if for some $d \in \mathbb{R}$, $\phi(i) \leq i^d$ for all $i$ sufficiently large. In this situation, we consider the number $\gamma(\phi) := \inf\left\{d : f(i) \leq i^d \text{ for } i \gg 0\right\}$. For a filtered $R$-module $M$, the Gelfand-Kirillov dimension is $\operatorname{GKdim} M := \gamma\left(\dim_k F_iM\right)$ . It follows, from subsection 8.6.18 in McConnell and Robson [McR], that $\operatorname{GKdim} M$ is independent of choice of generators $z_j$ and integers $w_j$, although it depends on the filtration of $R$ — see Proposition 6.2 for a discussion of this dependence.

Now, if $(\mathbf{u},\mathbf{v}) \in \operatorname{PR}(R)$ is not positive, then $\dim_k F_iM$ is infinite for all $i$. Thus, for $R$ to have a finite dimensional filtration, it is necessary and sufficient that the vector $(\mathbf{u},\mathbf{v})$ be positive. With this additional hypothesis, we can give a useful description of the function $i \mapsto \dim_k F_iM$. Recall that a function $\theta\colon \mathbb{Z} \to \mathbb{C}$ is called a quasi-polynomial if there exists a positive integer $p$ and polynomials $Q_j$ for $0 \leq j \leq p-1$ such that, for all $i \in \mathbb{Z}$, we have $\theta(i) = Q_j(i)$ where

$i = rp + j$ with $0 \leq j \leq p-1$. The degree of a quasi-polynomial is the maximum of degree of the polynomials $Q_j$. The next proposition is a small extension of the almost commutative results in section 8.4 of McConnell and Robson [McR].

**Proposition 3.1.** *Assume the vector* $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$ *is positive. If* $M$ *is a nonzero* $R$*-module with a good finite dimensional filtration such that* $\mathrm{gr}(M)$ *has Krull dimension* $d$*, then one has the following:*

(1) *There exists positive integers* $c_0, \dots, c_d$ *and* $Q(t) \in \mathbb{Z}[t, t^{-1}]$ *such that*
$$\sum_{i \geq 0} \big(\dim_k F_i M\big) \cdot t^i = \frac{Q(t)}{\prod_{j=0}^{d} (1 - t^{c_j})}\,, \text{ with } Q(1) > 0.$$
(2) *The function* $i \mapsto \dim_k F_i M$ *is a quasi-polynomial of degree* $d$*.*

*Proof.* Since $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$ is positive, $S$ is positively graded commutative $k$-algebra and Proposition 4.4.1 in Bruns and Herzog [BH] implies there are positive integers $c_1, \dots, c_d$ and $Q(t) \in \mathbb{Z}[t, t^{-1}]$ such that $\sum_{i \geq 0} \big(\dim_k \mathrm{gr}(M)_i\big) \cdot t^i = Q(t) / \prod_{j=1}^{d} (1 - t^{c_j})$ and $Q(1) > 0$. Hence, we have
$$\begin{aligned}\sum_{i \geq 0} \big(\dim_k F_i M\big) \cdot t^i &= \sum_{i \geq 0} \left( \sum_{j=0}^{i} \dim_k \mathrm{gr}(M)_j \right) \cdot t^i \\ &= \left( \frac{Q(t)}{\prod_{j=1}^{d} (1 - t^{c_j})} \right) \left( \frac{1}{1-t} \right),\end{aligned}$$
which proves part (1). Part (2) follows immediately from part (1) by applying Proposition 4.4.1 in Stanley [Sta]. □

The second part of this proposition clearly implies the following:

**Corollary 3.2.** *If the vector* $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$ *is positive and* $N$ *is a finitely generated graded* $S$*-module, then the Gelfand-Kirillov dimension and Krull dimension of* $N$ *are equal.* □

We now turn our attention to Rees modules and provide homomorphisms linking the submodules of $M$, $\widetilde{M}$ and $\mathrm{gr}(M)$. We always assume that the filtration on a submodule is the unique filtration induced by the module containing it. Because $R$ is left noetherian, good filtrations induce good filtrations on submodules.

**Proposition 3.3.** *Let* $x_0$ *be the canonical homogeneous element of degree* 1 *in Rees ring* $\widetilde{R}$*. If* $M$ *is a filtered* $R$*-module and* $\widetilde{M}$ *is the associated Rees module, one has the following:*

(1) *There exists a surjective homomorphism* $\pi_1 \colon \widetilde{M} \to M$ *such that* $\operatorname{Ker} \pi_1 = (1 - x_0) \cdot M$. *Moreover, for all submodules* $M' \subseteq M$, *one has* $\pi_1(\widetilde{M'}) = M'$.

(2) *There exists a surjective graded homomorphism* $\pi_0 \colon \widetilde{M} \to \operatorname{gr}(M)$ *such that* $\operatorname{Ker} \pi_0 = x_0 \cdot M$. *Furthermore,* $\pi_0$ *maps graded submodules of* $\widetilde{M}$ *to graded submodules of* $\operatorname{gr}(M)$ *and every graded submodule of* $\operatorname{gr}(M)$ *arises in this manner.*

*Proof.* (1) Every element $\tilde{z} \in \widetilde{M}$ can be written uniquely as a finite sum of homogeneous components; $\tilde{z} = \sum_{j=0}^{p} (\tilde{z})_{\ell_j}$ where $\ell_0 < \cdots < \ell_p$. Let $\pi_1 \colon \widetilde{M} \to M$ be defined by $\pi_1(\tilde{z}) = \sum_{j=0}^{p} (z)_{\ell_j}$ where $(z)_{\ell_j} \in F_{\ell_j} M$. The definition of the $\widetilde{R}$-module structure on $\widetilde{M}$ insures that $\pi_1$ is a $k$-module homomorphism and the image is an $R$-module. It is clearly surjective. Now, if $\sum_{j=0}^{p} (z)_{\ell_j} = 0$ then $\sum_{j=0}^{p} (\tilde{z})_{\ell_j} x_0^{\ell_p - \ell_j} = 0$. Hence, the element

$$\tilde{z} = \sum_{j=0}^{p} (\tilde{z})_{\ell_j} - \sum_{j=0}^{p} (\tilde{z})_{\ell_j} x_0^{\ell_p - \ell_j} = \sum_{j=0}^{p} \left( (\tilde{z})_{\ell_j} - (\tilde{z})_{\ell_j} x_0^{\ell_p - \ell_j} \right)$$

belongs to $(1 - x_0) \cdot \widetilde{M}$ and we have $\operatorname{Ker} \pi_1 \subseteq (1 - x_0) \cdot \widetilde{M}$. It is obvious from the definition of $\pi_1$ that we have $(1 - x_0) \cdot \widetilde{M} \subseteq \operatorname{Ker} \pi_1$ and $\pi_1(\widetilde{M'}) = M'$.

(2) For all $i \in \mathbb{Z}$, we have isomorphisms

$$\widetilde{M}_i / (x_0 \cdot \widetilde{M}_{i-1}) \cong F_i M / F_{i-1} M = \operatorname{gr}(M)_i \,.$$

Combining this maps gives the required isomorphism $\widetilde{M} / (x_0 \cdot \widetilde{M}) \cong \operatorname{gr}(M)$. Moreover, we have

$$\pi_0(\tilde{f} \tilde{z}) = f z + (x_0 \cdot M) = (f + x_0 \cdot R)(z + x_0 \cdot M) = \pi_0(\tilde{f}) \pi_0(\tilde{z})$$

and, thus, $\pi_0$ takes $\widetilde{R}$-modules to $\operatorname{gr}(R)$-modules. Finally, for a graded submodule $N$ of $\operatorname{gr}(M)$, consider the $\widetilde{R}$-submodule $L$ of $\widetilde{M}$ generated by the set $\pi_0^{-1}(N)$. To demonstrate that $\pi_0(L) = N$, it suffices to show $\pi_0(L) \subseteq N$. Every element of $L$ can be written in the form $\sum_{j=0}^{p} \tilde{f}_j \tilde{z}_j$ for some $\tilde{f}_j \in \widetilde{R}$ and $\tilde{z}_j \in \pi_0^{-1}(N)$. Applying $\pi_0$, we obtain

$$\pi_0 \left( \sum_{j=0}^{p} \tilde{f}_j \tilde{z}_j \right) = \sum_{j=0}^{p} \pi_0(\tilde{f}_j) \pi_0(\tilde{z}_j) = \sum_{j=0}^{p} f_j z_j$$

where $f_j \in \operatorname{gr}(A)$ and $z_j \in N$. Therefore, we have $\pi_0(L) \subseteq N$ which completes the proof. □

We next record a useful lemma; see Proposition 2.16 in Björk [Bj2].

**Lemma 3.4** (Björk). *Let $M$ be a filtered $R$-module. If $L$ be a graded submodule of $\widetilde{M}$, then the graded module $\widetilde{\pi_1(L)}$ contains $L$ and the quotient $\widetilde{\pi_1(L)}/L$ is an $x_0$-torsion module.* □

**Proposition 3.5.** *If $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$ is positive, $M$ is an $R$-module with a good finite dimensional filtration and $L$ is a graded submodule of $\widetilde{M}$, then one has the following:*

(1) $\mathrm{GKdim}\, L = \mathrm{GKdim}\, \widetilde{\pi_1(L)}$;
(2) $1 + \mathrm{GKdim}\, M = \mathrm{GKdim}\, \widetilde{M}$;
(3) $1 + \mathrm{GKdim}\, \pi_1(L) = \mathrm{GKdim}\, L$.

*Proof.* (1) Let $\phi(i) = \dim_k L_i$, $L' := \widetilde{\pi_1(L)}$ and $\psi(i) = \dim_k L'_i$. By Lemma 3.4, $L_i$ is a subvector space of $L'_i$ which implies $\phi(i) \leq \psi(i)$ and $\mathrm{GKdim}\, L \leq \mathrm{GKdim}\, L'$. On the other hand, Lemma 3.4 also states that the quotient $L'/L$ is an $x_0$-torsion module. Since $L'$ is a finitely generated module, there exists an integer $\ell$ such that $x_0^\ell \cdot L'_i \subseteq L_{i+\ell}$. Thus, we have $\psi(i) \leq \phi(i + \ell)$ which implies $\mathrm{GKdim}\, L' \leq \mathrm{GKdim}\, L$. Combining the two inequality yields the first part.

(2) The definition of Gelfand-Kirillov dimension implies

$$\begin{aligned} \mathrm{GKdim}\, M &= \gamma\left(\dim_k F_i M\right) \text{ and} \\ \mathrm{GKdim}\, \widetilde{M} &= \gamma\left(\sum_{j=0}^{i} \dim_k F_j M\right). \end{aligned}$$

However, a monotonically increasing function $\phi\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ and the function $\psi(i) = \sum_{j=0}^{i} \phi(j)$ are related by the equation $\gamma(\psi) = \gamma(\phi) + 1$ and this proves the second assertion.

(3) Applying part (2) gives $1 + \mathrm{GKdim}\, \pi_1(L) = \mathrm{GKdim}\, \widetilde{\pi_1(L)}$ and combining this part (1) yields the third assertion. □

We have the analogous result for submodules of $\widetilde{M}$ and $\mathrm{gr}(M)$.

**Proposition 3.6.** *Let $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$ be a positive vector and let $M$ be an $R$-module with a good filtration. If $L$ is a graded $\widetilde{R}$-submodule of $\widetilde{M}$ then $1 + \mathrm{GKdim}\, \pi_0(L) = \mathrm{GKdim}\, L$.*

*Proof.* We begin by observing the following:

$$\dim_k F_i\big(\pi_0(L)\big) = \sum_{j=0}^{i} \dim_k \frac{L_j}{x_0 \cdot L_{j-1}} = \dim_k L_i\,.$$

Next, notice that $\dim_k F_i L = \sum_{j=0}^{i} \dim_k L_j$. Since $x_0$ is a nonzero-divisor of degree 1 on $L$, we have $\dim_k L_i \leq \dim_k L_{i+1}$. Thus, applying

the formula $\gamma(\psi) = \gamma(\phi) + 1$ for a monotonically increasing function $\phi\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ and the function $\psi(i) = \sum_{j=0}^{i} \phi(j)$, completes the proof. $\square$

We now link the dimension of the irreducible components of the characteristic variety $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ to the Gelfand-Kirillov dimension of submodules of $M$.

**Theorem 3.7.** *Let $p$ be a nonnegative integer, let $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$ be a positive vector and let $M$ be a filtered $R$-module. If $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ has an irreducible component of dimension $p$ then there exists a submodule $M'$ of $M$ such that $\operatorname{GKdim} M' = p$.*

*Proof.* By definition, irreducible components of $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ correspond to minimal primes in the support of $\mathrm{gr}(M)$. Hence, if there exists an irreducible component of $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ of dimension $p$, we have a minimal prime $\mathfrak{p}$ in the support of $\mathrm{gr}(M)$ of dimension $p$. Each minimal prime $\mathfrak{p}$ in the support of $\mathrm{gr}(M)$ corresponds to a graded submodule of $\mathrm{gr}(M)$ of the form $\big(S/\mathfrak{p}\big)(j)$ for some $j \in \mathbb{Z}$ and Corollary 3.2 implies the Krull dimension of $\big(S/\mathfrak{p}\big)(j)$ is equal to its Gelfand-Kirillov dimension. Thus, we have a graded submodule of $\mathrm{gr}(M)$ with Gelfand-Kirillov dimension $p$. We complete this proof by showing that the following three conditions are equivalent:

(a) there exist a submodule $M'$ of $M$ with $\operatorname{GKdim} M' = p$;
(b) there exist a graded submodule $L$ of $\widetilde{M}$ with $\operatorname{GKdim} L = p + 1$;
(c) there exist a graded submodule $N$ of $\mathrm{gr}(M)$ with $\operatorname{GKdim} N = p$;

Indeed, we have:

(a) $\Rightarrow$ (b): By Proposition 3.5.2, the graded submodule $\widetilde{M'}$ of $\widetilde{M}$ has Gelfand-Kirillov dimension $p + 1$.
(b) $\Rightarrow$ (a): Follows immediately from Proposition 3.5.3.
(b) $\Rightarrow$ (c): Follows immediately from Proposition 3.6.
(c) $\Rightarrow$ (b): By Proposition 3.3, there exists a graded submodule $L$ of $\widetilde{M}$ such that $\pi_0(L) = N$.

$\square$

*Remark* 3.8. Since $\operatorname{GKdim} M = \operatorname{GKdim} \mathrm{gr}(M)$ for any $R$-module $M$, we see that the Gelfand-Kirillov dimension of $M$ is an upper bound on the Krull dimension of each irreducible component of $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$.

We point out that Theorem 3.7 can also be proven using homological methods; see Chapter 2 in Björk [Bj1] or Chapter III in Huishi and van Oystaeyen [HvO] for the techniques. However, the approach presented here is more elementary.

*Remark* 3.9. We have not used the fact that $R$ is an almost centralizing extension of $B$, so Proposition 3.3, Proposition 3.5, Proposition 3.6 and Theorem 3.7 hold a module $M$ with a good finite dimensional filtration over a filtered $k$-algebra.

## 4. Gröbner Basics

This section is devoted to the Gröbner basics of the skew polynomial ring $R$. Sturmfels' expression "Gröbner basics" describes a collection of ideas centering around initial ideals with respect to a vector $(\mathbf{u}, \mathbf{v})$ and term orders. We develop the algebraic aspects of this theory, generalizing results in commutative polynomial ring and Weyl algebra.

**Gröbner basis with respect to $(\mathbf{u}, \mathbf{v})$.** For $f = \sum \kappa_{\mathbf{a},\mathbf{b}} x^{\mathbf{a}} y^{\mathbf{b}}$ in $R$, the initial form (also called the principal symbol) of $f$ with respect to $(\mathbf{u}, \mathbf{v})$ is the element $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(f) = \sum_{\mathbf{u}\cdot\mathbf{a}+\mathbf{v}\cdot\mathbf{b}=\ell} \kappa_{\mathbf{a},\mathbf{b}} \bar{x}^{\mathbf{a}} \bar{y}^{\mathbf{b}}$ in $S$, where $\ell = \max\{\mathbf{u}\cdot\mathbf{a} + \mathbf{v}\cdot\mathbf{b} : \kappa_{\mathbf{a},\mathbf{b}} \neq 0\}$.

**Proposition 4.1.** *If $I$ is an $R$-ideal and $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$ then*

$$\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I) = k \cdot \left\{ \mathrm{in}_{(\mathbf{u},\mathbf{v})}(f) : f \in I \right\}$$

*is an $S$-ideal. Moreover, if $(\mathbf{u}, \mathbf{v})$ is an integer vector belonging to $\mathrm{PR}(R)$ then $\mathrm{gr}(I)$ is isomorphic to $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$.*

*Proof.* By definition, $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$ is closed under addition, so it is enough to show that it is closed under left multiplication by the elements $\bar{x}_1, \ldots, \bar{x}_m, \bar{y}_1, \ldots, \bar{y}_n$. Consider the element $f = \sum \kappa_{\mathbf{a},\mathbf{b}} x^{\mathbf{a}} y^{\mathbf{b}}$ in $I$. One clearly has

$$\bar{x}_i \cdot \mathrm{in}_{(\mathbf{u},\mathbf{v})}(f) = \mathrm{in}_{(\mathbf{u},\mathbf{v})} \left( \sum \kappa_{\mathbf{a},\mathbf{b}} x^{\mathbf{a}+\mathbf{e}_i} y^{\mathbf{b}} \right) = \mathrm{in}_{(\mathbf{u},\mathbf{v})}(x_i \cdot f)\,,$$

where $\mathbf{e}_i$ is the $i$-th standard basis vector. Similarly, because $(\mathbf{u}, \mathbf{v})$ belongs to $\mathrm{PR}(R)$, we obtain

$$\bar{y}_i \cdot \mathrm{in}_{(\mathbf{u},\mathbf{v})}(f) = \mathrm{in}_{(\mathbf{u},\mathbf{v})} \left( \sum \kappa_{\mathbf{a},\mathbf{b}} x^{\mathbf{a}} y^{\mathbf{b}+\mathbf{e}_i} + Q(x, y) \right) = \mathrm{in}_{(\mathbf{u},\mathbf{v})}(y_i \cdot f)\,,$$

where $Q(x, y) \in R$ has an initial form which is strictly smaller than $\mathrm{in}_{(\mathbf{u},\mathbf{v})} \left( \sum \kappa_{\mathbf{a},\mathbf{b}} x^{\mathbf{a}} y^{\mathbf{b}+\mathbf{e}_i} \right)$.

When $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R) \cap \mathbb{Z}^m \times \mathbb{Z}^n$, we note that

$$\{\mathrm{in}_{(\mathbf{u},\mathbf{v})}(f) : \; f \in I \text{ and } \deg_{(\mathbf{u},\mathbf{v})}(f) = i\}$$

is a complete set of representatives for the cosets of $\mathrm{gr}(I)_i$. Hence, there exists a bijective set map between $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$ and $\mathrm{gr}(I)$ and one easily verifies that the $S$-module structure of $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$ and $\mathrm{gr}(I)$ agree under this correspondence. $\square$

The $S$-ideal $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$ is called the initial ideal of the $R$-ideal $I$ with respect to the vector $(\mathbf{u},\mathbf{v})$. A finite subset $\mathcal{G}$ of $R$ is a Gröbner basis of $I$ with respect to $(\mathbf{u},\mathbf{v})$ if $I$ is generated by $\mathcal{G}$ and $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$ is generated by the initial forms $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(\mathcal{G}) = \big\{\mathrm{in}_{(\mathbf{u},\mathbf{v})}(g) : g \in \mathcal{G}\big\}$.

**Gröbner bases with respect to $\prec$.** A total ordering $\prec$ on the set of standard monomials $\big\{x^{\mathbf{a}}y^{\mathbf{b}} : (\mathbf{a},\mathbf{b}) \in \mathbb{N}^m \times \mathbb{N}^n\big\}$ in $R$ is called a multiplicative order when the following three conditions hold:

(M1) $x^{\mathbf{a}} \prec x_i y_j$ for all monomials $x^{\mathbf{a}}$ appearing in $Q^1_{i,j}(x)$;

(M2) $x^{\mathbf{a}} y_\ell \prec y_i y_j$ for all monomials $x^{\mathbf{a}} y_\ell$ appearing in $Q^2_{i,j}(x,y)$;

(M3) $x^{\mathbf{a}}y^{\mathbf{b}} \prec x^{\mathbf{a}'}y^{\mathbf{b}'} \Rightarrow x^{\mathbf{a}+\mathbf{c}}y^{\mathbf{b}+\mathbf{d}} \prec x^{\mathbf{a}'+\mathbf{c}}y^{\mathbf{b}'+\mathbf{d}}\ \forall\, (\mathbf{c},\mathbf{d}) \in \mathbb{N}^m \times \mathbb{N}^n$.

A multiplicative order $\prec$ is called a term order if $1 = x^0 y^0$ is the smallest element of $\prec$. A multiplicative order which is not a term order has infinite strictly decreasing chains but a term order does not. For information on frequently used term orders such as lexicographic order or reverse lexicographic order, see chapter 15 in Eisenbud [Eis].

*Remark* 4.2. Conditions (M1) and (M2) correspond directly to the relations $(R1)$ and $(R2)$ in the definition of $R$. Without these assumptions the order would not be compatible with multiplication; that is we would not have $\mathrm{in}_\prec(f \cdot f') = \mathrm{in}_\prec(f)\,\mathrm{in}_\prec(f')$.

Fix a multiplicative order $\prec$. The initial monomial of $f \in R$ is the monomial $\bar{x}^{\mathbf{a}}\bar{y}^{\mathbf{b}} \in S$ such that $x^{\mathbf{a}}y^{\mathbf{b}}$ is the $\prec$-largest monomial appearing in the standard expansion of $f$ in $R$. For an $R$-ideal $I$, the initial ideal $\mathrm{in}_\prec(I)$ is the monomial ideal in $S$ generated by $\{\mathrm{in}_\prec(f) :\ f \in I\}$. A finite subset $\mathcal{G}$ of $R$ is a Gröbner basis of $I$ with respect to $\prec$ if $I$ is generated by $\mathcal{G}$ and $\mathrm{in}_\prec(I)$ is generated by $\mathrm{in}_\prec(\mathcal{G}) = \{\mathrm{in}_\prec(g) : g \in \mathcal{G}\}$. The Gröbner basis is called reduced if, for any two distinct elements $g$, $g' \in \mathcal{G}$, the exponent vector of $\mathrm{in}_\prec(g)$ is componentwise larger than any exponent vector appearing in the normally order expression of $g'$ in $R$.

**Comparing $\prec$ and $(\mathbf{u},\mathbf{v})$.** We have two different notions of Gröbner basis in $R$; one for degree vectors $(\mathbf{u},\mathbf{v})$ and one for multiplicative monomial orders. Proposition 4.4 below relates these two notions. Before giving this relation, we collect some preliminary results. Let $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$ and let $\prec$ be any term order. The multiplicative monomial order $\prec_{(\mathbf{u},\mathbf{v})}$ is defined as follows:

$$x^{\mathbf{a}'}y^{\mathbf{b}'} \prec x^{\mathbf{a}}y^{\mathbf{b}} \quad \begin{array}{l} \text{if and only if} \\ \big(\mathbf{u}\cdot(\mathbf{a}-\mathbf{a}') + \mathbf{v}\cdot(\mathbf{b}-\mathbf{b}') > 0\big) \\ \text{or } \left( \begin{array}{l} \mathbf{u}\cdot(\mathbf{a}-\mathbf{a}') + \mathbf{v}\cdot(\mathbf{b}-\mathbf{b}') = 0 \\ \text{and } x^{\mathbf{a}'}y^{\mathbf{b}'} \prec x^{\mathbf{a}}y^{\mathbf{b}} \end{array} \right). \end{array}$$

Note that $\prec_{(\mathbf{u},\mathbf{v})}$ is a term order if and only if $(\mathbf{u},\mathbf{v})$ is a non-negative vector.

**Lemma 4.3.** *If $I$ is an $R$-ideal and $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$, then*

$$\mathrm{in}_{\prec}\left(\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)\right) = \mathrm{in}_{\prec_{(\mathbf{u},\mathbf{v})}}(I)\,.$$

*Proof.* See Proposition 1.8 is Sturmfels [Stu]. □

**Proposition 4.4.** *Let $I$ be any $R$-ideal, $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$ and let $\prec$ be any term order. If $\mathcal{G}$ a Gröbner basis for $I$ with respect to $\prec_{(\mathbf{u},\mathbf{v})}$, then one has*

(1) *the set $\mathcal{G}$ is a Gröbner basis for $I$ with respect to $(\mathbf{u},\mathbf{v})$;*
(2) *the set $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(\mathcal{G}) = \left\{\mathrm{in}_{(\mathbf{u},\mathbf{v})}(g) : g \in \mathcal{G}\right\}$ is a Gröbner basis for $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$ with respect to $\prec$;*
(3) *if $\mathcal{G}$ is the reduced Gröbner basis for $I$ with respect to $\prec_{(\mathbf{u},\mathbf{v})}$, then $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(\mathcal{G})$ is also reduced.*

*Proof.* Parts (1) and (2) are analogous to Theorem 1.1.6 in Saito, Sturmfels and Takayama [SST]. Part (3) follows from the fact that the exponent vectors appearing in $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(g)$ for $g \in \mathcal{G}$ form a subset of the exponent vectors appearing the standard expression of $g$ in $R$. □

**The number of initial ideals.** Although there are infinitely many different term orders when $m + n \geq 2$, this does not lead to an infinite number of distinct initial ideals.

**Theorem 4.5.** *An $R$-ideal $I$ has only finitely many distinct initial ideals $\mathrm{in}_{\prec}(I)$ where $\prec$ is a term order.*

*Proof.* See Theorem 1.2 in Sturmfels [Stu]. □

**Homogenization.** A multiplicative order $\prec$ on $R$ lifts to a multiplicative order $<$ on $\widetilde{R}$ by the following convention:

$$x_0^{a_0'} x^{\mathbf{a}'} y^{\mathbf{b}'} < x_0^{a_0} x^{\mathbf{a}} y^{\mathbf{b}} \quad \begin{array}{l} \text{if and only if} \\ \left(a_0' - a_0 > 0\right) \\ \text{or} \left( \begin{array}{c} a_0' - a_0 = 0 \\ \text{and } x^{\mathbf{a}'} y^{\mathbf{b}'} \prec x^{\mathbf{a}} y^{\mathbf{b}} \end{array} \right). \end{array}$$

Note that $\prec$ is a term order if and only if $<$ is a term order.

**Proposition 4.6.** *Let $\prec$ be a multiplicative order on $R$ and let $\widetilde{I}$ be the homogenization of an $R$-ideal $I$ with respect to $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$. If $\widetilde{\mathcal{G}}$ is a Gröbner basis for $\widetilde{I}$ with respect to $<$ then its dehomogenization $\mathcal{G}$ is a Gröbner basis for $I$ with respect to $\prec$.*

*Proof.* Since $\widetilde{I}|_{x_0=1} = I$, the set $\mathcal{G}$ generates $I$ if and only if $\widetilde{\mathcal{G}}$ generates $\widetilde{I}$. Thus, it suffices to study the initial ideals. If $h \in \mathrm{in}_{\prec}(I)$ then we have $\tilde{h} \in \mathrm{in}_{<}(\widetilde{I})$. Since $\widetilde{\mathcal{G}}$ is a Gröbner basis, it follows that $\tilde{h} = \mathrm{in}_{<}(\tilde{f}\tilde{g})$ for some $\tilde{f} \in \widetilde{R}$ and $\tilde{g} \in \widetilde{\mathcal{G}}$. Dehomogenizing, we obtain $h = \tilde{h}|_{x_0=1} = \mathrm{in}_{\prec}(\tilde{f}|_{x_0=1} \cdot \tilde{g}|_{x_0=1})$ which implies $h \in \mathrm{in}_{\prec}(\mathcal{G})$ as required. □

Fix an $R$-ideal $I$. Two degree vectors $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{u}', \mathbf{v}')$ in $\mathrm{PR}(R)$ are equivalent with respect to $I$ if $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I) = \mathrm{in}_{(\mathbf{u}',\mathbf{v}')}(I)$. We denote the equivalence class of vectors $(\mathbf{u}, \mathbf{v})$ with respect to $I$ by $C_I[(\mathbf{u}, \mathbf{v})]$. We define the Gröbner region $\mathrm{GR}(I)$ to be the set of all $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$ such that $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I) = \mathrm{in}_{(\mathbf{u}',\mathbf{v}')}(I)$ for some positive vector $(\mathbf{u}', \mathbf{v}') \in \mathrm{PR}(R)$.

**Proposition 4.7.** *Suppose that $R$ is a graded $k$-algebra with respect to a positive vector $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$. If $I$ is a homogeneous $R$-ideal, then we have $\mathrm{GR}(I) = \mathrm{PR}(R)$.*

Notice that, for any $R$-ideal $I$, the Rees ring $\widetilde{R}$ associated to a positive vector $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$ and $\widetilde{I}$ satisfy the hypothesis.

*Proof.* See Proposition 1.12 in Sturmfels [Stu]. □

A finite subset $\mathcal{U}$ of an $R$-ideal $I$ is called a universal Gröbner basis if $\mathcal{U}$ is simultaneously a Gröbner basis of $I$ with respect to all vectors $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$. This definition is different than the one found Sturmfels [Stu]; Sturmfels' considers only vectors $(\mathbf{u}, \mathbf{v})$ in the Gröbner region $\mathrm{GR}(I)$. Proposition 4.7 shows that these two different notions of a universal Gröbner basis agree for homogeneous ideals in a graded ring.

**Corollary 4.8.** *Every $R$-ideal $I$ has a finite universal Gröbner basis.*

*Proof.* Consider homogenization $\widetilde{I}$ of $I$ with respect to a positive vector $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$. By Theorem 4.5 there exists only finitely many distinct reduced Gröbner basis for $\widetilde{I}$ with respect to term orders. Let $\widetilde{\mathcal{G}}$ be their union. Fix a term order $\prec$ on $R$ and let $<_{(\mathbf{u},\mathbf{v})}$ denote the multiplicative order on $\widetilde{R}$ obtained from the multiplicative order $\prec_{(\mathbf{u},\mathbf{v})}$ on $R$. By construction, $\widetilde{\mathcal{G}}$ is a Gröbner basis with respect to $<_{(\mathbf{u},\mathbf{v})}$ when $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$ is positive. Applying Proposition 4.3 and Proposition 4.7, we see that $\widetilde{\mathcal{G}}$ is, in fact, a Gröbner basis with respect to $<_{(\mathbf{u},\mathbf{v})}$ when $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$. If $\mathcal{G}$ is the dehomogenization of $\widetilde{\mathcal{G}}$, then Proposition 4.6 implies that $\mathcal{G}$ is a Gröbner basis of $I$ for $\prec_{(\mathbf{u},\mathbf{v})}$ where $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$. Finally, Proposition 4.4.1 shows that $\mathcal{G}$ is a universal Gröbner basis for $I$. □

*Remark* 4.9. Since they are not required for our applications, we have omitted a discussion of the computational aspects; most notably, the division algorithm and Buchberger's criterion.

## 5. The Gröbner Fan

The purpose of this section is to describe the initial ideals and the natural adjacency relations among them. For a commutative polynomial ring, Mora and Robbiano's Gröbner fan [MR] accomplishes this goal; Assi, Castro-Jiménez and Granger [ACG] have extended this to the Weyl algebra and we generalize it to the skew polynomial ring $R$. Our setting has the advantage that the commutative polynomial ring, Weyl algebra and homogenized Weyl algebra are all done at once, shortening the treatment in Saito, Sturmfels and Takayama [SST].

The next proposition shows that Gröbner bases with respect to vectors $(\mathbf{u},\mathbf{v})$ generalize those with respect to term orders.

**Proposition 5.1.** *Let $I$ be an $R$-ideal. For any term order $\prec$ there exists a positive vector $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$ such that $\mathrm{in}_{\prec}(I) = \mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$.*

*Proof.* See Proposition 2.1.5 in Saito, Sturmfels and Takayama [SST]. $\square$

We prove a key tool in the construction of the Gröbner fan.

**Proposition 5.2.** *Let $I$ be an $R$-ideal, let $(\mathbf{u}',\mathbf{v}')$ belong to $\mathrm{PR}(R)$ and let $(\mathbf{u},\mathbf{v})$ be a vector in $\mathrm{PR}(S) = \mathbb{R}^m \times \mathbb{R}^n$. If $\varepsilon > 0$ is sufficiently small, then one has $\mathrm{in}_{(\mathbf{u},\mathbf{v})}\big(\mathrm{in}_{(\mathbf{u}',\mathbf{v}')}(I)\big) = \mathrm{in}_{(\mathbf{u}'+\varepsilon\mathbf{u},\mathbf{v}'+\varepsilon\mathbf{v})}(I)$.*

*Proof.* Let $\prec$ be any term order and let $\prec_\varepsilon$ be the multiplicative monomial order on $R$ defined as follows:

$$x^{\mathbf{a}'}y^{\mathbf{b}'} \prec_\varepsilon x^{\mathbf{a}}y^{\mathbf{b}} \quad \text{if and only if} \quad \begin{array}{l} \big((\mathbf{u}'+\varepsilon\mathbf{u},\mathbf{v}'+\varepsilon\mathbf{v})\cdot(\mathbf{a}-\mathbf{a}',\mathbf{b}-\mathbf{b}') > 0\big) \\ \text{or } \left(\begin{array}{l} (\mathbf{u}'+\varepsilon\mathbf{u},\mathbf{v}'+\varepsilon\mathbf{v})\cdot(\mathbf{a}-\mathbf{a}',\mathbf{b}-\mathbf{b}') = 0 \\ \text{and } (\mathbf{u}',\mathbf{v}')\cdot(\mathbf{a}-\mathbf{a}',\mathbf{b}-\mathbf{b}') > 0 \end{array}\right) \\ \text{or } \left(\begin{array}{l} (\mathbf{u}'+\varepsilon\mathbf{u},\mathbf{v}'+\varepsilon\mathbf{v})\cdot(\mathbf{a}-\mathbf{a}',\mathbf{b}-\mathbf{b}') = 0, \\ (\mathbf{u}',\mathbf{v}')\cdot(\mathbf{a}-\mathbf{a}',\mathbf{b}-\mathbf{b}') = 0 \\ \text{and } x^{\mathbf{a}'}y^{\mathbf{b}'} \prec x^{\mathbf{a}}y^{\mathbf{b}}) \end{array}\right). \end{array}$$

Next, fix a universal Gröbner basis $\mathcal{U}$ for $I$ and choose $\varepsilon$ small enough so that the following assertions hold: $(\mathbf{u}'+\varepsilon\mathbf{u},\mathbf{v}'+\varepsilon\mathbf{v}) \in \mathrm{PR}(R)$ and for all elements $g$ in $\mathcal{U}$, the standard form of $g$ breaks into four pieces $g(x,y) = g_0(x,y) + g_1(x,y) + g_2(x,y) + g_3(x,y)$ such that $\mathrm{in}_{\prec_\varepsilon}(g) = g_0(\bar{x},\bar{y})$, $\mathrm{in}_{(\mathbf{u}'+\varepsilon\mathbf{u},\mathbf{v}'+\varepsilon\mathbf{v})}(g) = g_0(\bar{x},\bar{y}) + g_1(\bar{x},\bar{y})$, and $\mathrm{in}_{(\mathbf{u}',\mathbf{v}')}(g) = g_0(\bar{x},\bar{y}) +$

$g_1(\bar{x}, \bar{y}) + g_2(\bar{x}, \bar{y})$. In particular, we have

$$(\dagger) \qquad \mathrm{in}_{(\mathbf{u},\mathbf{v})}\left(g_0(x,y) + g_1(x,y) + g_2(x,y)\right) = g_0(\bar{x}, \bar{y}) + g_1(\bar{x}, \bar{y})\,.$$

Since $\mathcal{U}$ is a Gröbner basis with respect to $\prec_\varepsilon$, Proposition 4.4 provides two additional Gröbner bases in the polynomial ring $S$:

(i) The initial forms $g_0(\bar{x}, \bar{y}) + g_1(\bar{x}, \bar{y})$ for $g \in \mathcal{U}$ are a Gröbner basis for the initial ideal $\mathrm{in}_{(\mathbf{u}'+\varepsilon\mathbf{u},\mathbf{v}'+\varepsilon\mathbf{v})}(I)$ with respect to $\prec_\varepsilon$.

(ii) The initial forms $g_0(\bar{x}, \bar{y}) + g_1(\bar{x}, \bar{y}) + g_2(\bar{x}, \bar{y})$ for $g \in \mathcal{U}$ are a Gröbner basis for the initial ideal $\mathrm{in}_{(\mathbf{u}',\mathbf{v}')}(I)$ with respect to $\prec_\varepsilon$.

Now, the definition of $\prec_\varepsilon$, statement (ii) and Proposition 4.4.2 imply that the polynomials $g_0(\bar{x}, \bar{y}) + g_1(\bar{x}, \bar{y}) + g_2(\bar{x}, \bar{y})$ for $g \in \mathcal{U}$ are a Gröbner basis for the ideal $\mathrm{in}_{(\mathbf{u}',\mathbf{v}')}(I)$ with respect the vector $(\mathbf{u}, \mathbf{v})$. Moreover, $(\dagger)$ indicates that the polynomials $g_0(\bar{x}, \bar{y}) + g_1(\bar{x}, \bar{y})$ for $g \in \mathcal{U}$ generate the ideal $\mathrm{in}_{(\mathbf{u},\mathbf{v})}\left(\mathrm{in}_{(\mathbf{u}',\mathbf{v}')}(I)\right)$ and therefore statement (ii) completes the proof. □

We are now in a position to give a description of the equivalence classes $C_I[(\mathbf{u}, \mathbf{v})]$.

**Proposition 5.3.** *Let $I$ be an $R$-ideal, let $\prec$ be a term order and let $(\mathbf{u}, \mathbf{v})$ belong to $\mathrm{GR}(I)$. If $\mathcal{G}$ is the reduced Gröbner basis of $I$ with respect to $\prec_{(\mathbf{u},\mathbf{v})}$, then one has*

$$(\ddagger) \quad C_I[(\mathbf{u}, \mathbf{v})] = \left\{(\mathbf{u}', \mathbf{v}') \in \mathrm{GR}(I) : \mathrm{in}_{(\mathbf{u},\mathbf{v})}(g) = \mathrm{in}_{(\mathbf{u}',\mathbf{v}')}(g) \ \forall\ g \in \mathcal{G}\right\}$$

*and, hence, each equivalence class $C_I[(\mathbf{u}, \mathbf{v})]$ is a relatively open rational convex polyhedral cone.*

*Proof.* See Proposition 2.3 in Sturmfels [Stu]. □

We end with the main result of this section.

**Theorem 5.4.** *For $I$ an $R$-ideal, the finite set*

$$\mathrm{GF}(I) := \left\{\overline{C_I[(\mathbf{u}, \mathbf{v})]} : \text{ for all } (\mathbf{u}, \mathbf{v}) \in \mathrm{GR}(I)\right\}$$

*forms a fan, called the Gröbner fan of $I$.*

*Proof.* Given the above lemmas and propositions, the proof is now identical to Proposition 2.4 in Sturmfels [Stu]. □

## 6. Bounds on the Irreducible Components

This section contains the proofs of the main results of this paper. We use the Gröbner fan is to show that the Gelfand-Kirillov dimension of $M$ is independent of the positive vector $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$. We start by stating an elementary lemma from commutative algebra – see Lemma 2.2.2 in Saito, Sturmfels and Takayama [SST].

**Lemma 6.1.** *If $J$ is any ideal in $S$ and $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^m \times \mathbb{R}^n$ then one has the following:*

(1) $\operatorname{Kdim} \operatorname{in}_{(\mathbf{u},\mathbf{v})}(J) \leq \operatorname{Kdim} J$.
(2) *If $(\mathbf{u}, \mathbf{v})$ is positive, then* $\operatorname{Kdim} \operatorname{in}_{(\mathbf{u},\mathbf{v})}(J) = \operatorname{Kdim} J$. □

Making use of the Gröbner fan, we next study effect that varying the filtration of $R$ has on the Gelfand-Kirillov dimension of an $R$-module.

**Proposition 6.2.** *If $M$ is a finitely generated $R$-module. then the Gelfand-Kirillov dimension of $M$ is independent of the positive vector $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$.*

Recall that $M$ has a good finite dimensional filtration if and only if the vector $(\mathbf{u}, \mathbf{v})$ is positive.

*Proof.* By subsection 8.6.5 in McConnell and Robson [McR], we know that if $M$ has a good finite dimensional filtration, then $\operatorname{GKdim} M = \operatorname{GKdim} \operatorname{gr}(M)$. Moreover, the Gelfand-Kirillov dimension and Krull dimension of $\operatorname{gr}(M)$ are equal by Corollary 3.2. Since the Krull dimension of a finitely generated module is the Krull dimension of its support, it suffices to consider ideals. In particular, one reduces to proving that, for an $R$-ideal $I$, the initial ideal $\operatorname{in}_{(\mathbf{u},\mathbf{v})}(I)$ is independent of the choice of positive vector $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R)$.

We prove this statement by constructing a homotopy or "Gröbner walk" between any two initial ideals. Let $(\mathbf{u}_1, \mathbf{v}_1)$ and $(\mathbf{u}_2, \mathbf{v}_2)$ be two positive vectors in $\mathrm{PR}(R)$. We claim that $\operatorname{Kdim} \operatorname{in}_{(\mathbf{u}_1,\mathbf{v}_1)}(I) = \operatorname{Kdim} \operatorname{in}_{(\mathbf{u}_2,\mathbf{v}_2)}(I)$. Moreover, Proposition 5.3 implies that each equivalence class $C_I[(\mathbf{u}_2, \mathbf{v}_2)]$ is a convex cone and, thus, we have $\operatorname{in}_{(\mathbf{u}_2,\mathbf{v}_2)}(I) = \operatorname{in}_{(r\mathbf{u}',r\mathbf{v}')}(I)$ for any positive $r \in \mathbb{R}$. Hence, we may, if necessary, replace $(\mathbf{u}_2, \mathbf{v}_2)$ by a positive scalar multiple, to guarantee that $(\mathbf{u}' - \mathbf{u}, \mathbf{v}' - \mathbf{v})$ is a positive vector. It follows that $(1-r) \cdot (\mathbf{u}_1, \mathbf{v}_1) + r \cdot (\mathbf{u}_2, \mathbf{v}_2)$ is a positive vector and belongs to $\mathrm{GR}(I)$ for all $r \in [0, 1]$. Define $J_r$ to be the $R$-ideal $\operatorname{in}_{(1-r)\cdot(\mathbf{u}_1,\mathbf{v}_1)+r\cdot(\mathbf{u}_2,\mathbf{v}_2)}(I)$. Since the line segment from $(\mathbf{u}_1, \mathbf{v}_1)$ to $(\mathbf{u}_2, \mathbf{v}_2)$ intersects finitely many distinct walls of the Gröbner fan, there are real numbers $0 = r_0 < r_1 < \cdots < r_\ell = 1$ such that the ideal $J_r$ remains unchanged as the parameter $r$ ranges inside the open interval $(r_j, r_{j+1})$; we denote this ideal by $J_{(r_j,r_{j+1})}$. By Proposition 5.2, we have $\operatorname{in}_{(\mathbf{u}_2,\mathbf{v}_2)}(J_{r_j}) = J_{(r_j,r_{j+1})} = \operatorname{in}_{(\mathbf{u}_1,\mathbf{v}_1)}(J_{r_{j+1}})$. By applying Lemma 6.1 2, we see that $\operatorname{Kdim} J_{r_j} = \operatorname{Kdim} J_{(r_j,r_{j+1})} = \operatorname{Kdim} J_{r_{j+1}}$. Combining these equalities for $0 \leq j < \ell$ completes the proof. □

We are now ready to prove:

**Theorem 6.3.** *Let $p$ be a nonnegative integer and let $M$ be a finitely generated $R$-module. If $\operatorname{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ has an irreducible component of*

*dimension $p$ for some $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$, then there is a submodule $M'$ of $M$ such that* $\operatorname{GKdim} M' = p$ *when $M'$ is equipped with a good finite dimensional filtration.*

*Proof.* Fix a positive vector in $\mathrm{PR}(R)$ and let $\widetilde{R}$ and $\widetilde{M}$ be associated Rees ring and module. Now, suppose $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ has an irreducible component of dimension $p$. Since

$$\operatorname{Ann}_S\big(\operatorname{gr}(M)\big) = \operatorname{gr}\big(\operatorname{Ann}_R(M)\big) = \operatorname{in}_{(\mathbf{u},\mathbf{v})}\big(\operatorname{Ann}_R(M)\big),$$

this means that the $S$-ideal $\operatorname{in}_{(\mathbf{u},\mathbf{v})}\big(\operatorname{Ann}_R(M)\big)$ has a minimal prime $\mathfrak{p}$ of dimension $p$. By Proposition 4.4 and Proposition 4.6, we have

$$\operatorname{in}_{(\mathbf{u},\mathbf{v})}\big(\operatorname{Ann}_R(M)\big) = \operatorname{in}_{(1,\mathbf{u},\mathbf{v})}\big(\operatorname{Ann}_{\widetilde{R}}(\widetilde{M})\big)\Big|_{\bar{x}_0=1}.$$

Because $\bar{x}_0$ is a nonzero-divisor on $S[\bar{x}_0]$, $\mathfrak{p}$ pulls back to a minimal prime of dimension $p+1$ over $\operatorname{in}_{(1,\mathbf{u},\mathbf{v})}\big(\operatorname{Ann}_{\widetilde{R}}(\widetilde{M})\big)$. Furthermore, there exists, by Proposition 4.7, a positive vector $(u_0,\mathbf{u}',\mathbf{v}') \in \mathrm{PR}(\widetilde{R})$ such that $\operatorname{in}_{(1,\mathbf{u},\mathbf{v})}\big(\operatorname{Ann}_{\widetilde{R}}(\widetilde{M})\big) = \operatorname{in}_{(u_0,\mathbf{u}',\mathbf{v}')}\big(\operatorname{Ann}_{\widetilde{R}}(\widetilde{M})\big)$. It follows, by Theorem 3.7, that $\widetilde{M}$ has a submodule $L$ such that $\operatorname{GKdim} L = p+1$. Finally, Proposition 3.5 implies $\pi_0(L)$ is a submodule of $M$ with Gelfand-Kirillov dimension $p$. ☐

Using this theorem, we prove the main results of this paper.

*Proof of Theorem 1.2.* If the characteristic variety $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ has an irreducible component of dimension strictly less than $p$, then Theorem 6.3 implies that there exists a submodule $M'$ of $M$ with Gelfand-Kirillov dimension strictly less than $p$. However, this contradicts our hypothesis. ☐

We recall Bernstein's inequality which is also called the Weak Fundamental Theorem of Algebraic Analysis—see Section 1.4 in Björk [Bj1] for two distinct proofs.

**Theorem 6.4** (Bernstein)**.** *Let $k$ be a field of characteristic zero. If $A_n(k)$ has the standard filtration $(\mathbf{1},\mathbf{1})$ and $M$ is a finitely generated $A_n$-module, then one has* $\operatorname{GKdim} M \geq n$. ☐

*Proof of Theorem 1.1.* This is immediate from Theorem 1.2 and Theorem 6.4. ☐

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